PHYSICAL REVIEW E VOLUME 52, NUMBER 1 JULY 1995

Geometric properties of isovorticity surfaces in magnetohydrodynamic turbulence

Vincenzo Carbone
Dipartimento di Fisica, Università della Calabria, 87036 Roges di Rende, Cosenza, Italy
(Received 25 July 1994; revised manuscript received 21 February 1995)

The stretching effect of vorticity in magnetohydrodynamic turbulence spontaneously generates flat or elongated structures that are unstable with respect to reconnecting (tearing-mode-type) perturbations. The instability could develop faster than the nonlinear energy cascade in a range of scale lengths whose extension depends on the Reynolds number. We argue that, if the instability is effectively at work, the fractal dimension of the isovorticity density surface in the above mentioned range is of the order of $D = 2.31$, in contrast to larger scales where $D = 2.75$.

PACS number(s): 47.65.+a, 47.27.−i, 52.35.Py

Small-scale stretching in fully developed turbulence causes both passive scalar as the temperature field and active field as the vorticity to concentrate in small regions of space. In general the level surfaces, say the surfaces of isoconcentration of the fields, assume a very complicated shape and the surface area has a fractal dimension D which is related to the scaling properties of the turbulent fields and depends on the length scale ℓ. The geometric properties of level surfaces for the temperature field and the vorticity in hydrodynamic turbulence have been recently studied both from a theoretical and an experimental point of view [1–3]. It has been found that, for scales ℓ larger than an inner scale ℓ_*, a level set appears as a smooth surface of dimension $D = 8/3$. This is true also in magnetohydrodynamic (MHD) turbulence [4], apart from the fact that in this case the dimension of the level sets of the current density is found to be higher, say $D = 11/4$, in agreement with the results of a numerical simulation of two-dimensional (2D) MHD turbulence [5].

Owing to the stretching effect, the small scales in MHD turbulence are unstable with respect to the development of reconnecting perturbations of the tearing mode type [6–9]. Actually the occurrence of the tearing instability in MHD turbulence is a matter of debate. However, if the instability was effectively at work in MHD turbulence, the small scales of MHD turbulence are the natural place in which the local energy transfer due to the nonlinear interactions could compete with the nonlocal energy transfer due to the tearing instability [7]. Since the small-scale dynamics of MHD turbulence is changed by this competition, we conjecture the possibility that, due to this effect, the dimension of the isovorticity surfaces could be changed to $D = 37/16$ in a range of scales intermediate between ℓ and ℓ_*.

The incompressible MHD equations can be written in terms of the Elsässer variables $\vec{Z}^{\pm} = \vec{v} \pm \vec{B}/(4\pi\rho)^{1/2}$,

$$\frac{\partial \vec{Z}^{\pm}}{\partial t} + \left(\vec{Z}^{\mp} \cdot \vec{\nabla}\right) \vec{Z}^{\pm} = -\frac{1}{\rho}\vec{\nabla} P + \nu \nabla^2 \vec{Z}^{\pm} . \quad (1)$$

The vectors $\vec{v}$ and $\vec{B}$ represent, respectively, the velocity field and the magnetic field, ρ is the constant plasma mass density, and P is the total (magnetic plus kinetic) pressure. Finally, ν represents the kinematic viscosity, assumed to be equal to the resistivity, in order not to worry about coupling between the modes $\vec{Z}^+$ and $\vec{Z}^-$ due to the dissipative terms. Generally [10] it is assumed that the nonlinear interactions are local, that is, they occur only between fluctuations $Z_\ell^{\pm} = \vec{e} \cdot [\vec{Z}^{\pm}(\vec{r} + \vec{\ell}) - \vec{Z}^{\pm}(\vec{r})]$ ($\vec{e}$ represents the longitudinal unitary vector) at the same length scale $\ell = |\vec{\ell}|$ in the inertial range ($\ell_d \ll \ell \ll L$, where ℓ_d is the dissipative scale and L is the largest scale). The main feature which distinguishes the MHD turbulence from the ordinary fluid turbulence is the Alfvén effect [10], that is, the advection of small-scale structures by waves in the large ones. This effect modifies the characteristic time needed to realize the nonlinear energy cascade and, as a consequence, the energy transfer rate per unit mass [10] is different from the usual Kolmogorov relation

$$\epsilon^{\pm} \sim \frac{(Z_\ell^{\pm})^2 (Z_\ell^{\mp})^2}{c_A \ell} \quad (2)$$

($\vec{c}_A = \vec{B}_0/\sqrt{4\pi\rho}$ is the Alfvén velocity relative to the inhomogeneous magnetic field $\vec{B}_0$ of the scale L). Relation (2) relates the stochastic variables $Z_\ell^{\pm}$ and $\epsilon_\ell^{\pm}$. The MHD equations (1) are scale invariant providing $\ell \to \ell\lambda^{-1}$ and $\vec{Z}^{\pm} \to \vec{Z}^{\pm}\lambda^h$ ($\lambda > 0$ and h is a parameter) [11]. Since Z_ℓ/ℓ^h is invariant (for $\ell \to 0$) we expect a scaling law where $Z_\ell^{\pm} \sim \ell^h$. The requirement that the energy transfer rate is independent from the length scale fixes the Kraichnan value $h = h^* = 1/4$. If intermittency is taken into account [12,11] a whole set of Hölder exponents h must be taken into account, so that the scaling exponents ξ_q of the q th order structure functions $\langle (Z_\ell^{\pm})^q \rangle \sim \ell^{\xi_q}$ [13] (brackets represent spatial averages), in terms of the set of generalized dimensions $\widehat{D}_q$ introduced by Hentschel and Procaccia [14], are given by [11]

$$\xi_q = \frac{q}{4} + \left(\frac{q}{4} - 1\right)(\widehat{D}_{q/4} - 1) . \quad (3)$$

It is generally well established (see Ref. [7], and references therein) that MHD incompressible turbulence, for a wide range of initial conditions, tends toward the steady state of Eq. (1) where

$$\vec{v} = \pm \frac{\vec{B}}{\sqrt{4\pi\rho}}$$

(dynamical alignment regime). The sign depends on the initial conditions. Due to the Alfvén effect [10], the dynamical alignment does not occur when $\vec{k}\cdot\vec{B}_0 = 0$ ($\vec{k}$ denotes the wave vector of a perturbation). To investigate what happens around these points, we assume that the process of dynamical alignment has taken place almost everywere in the fluid and that the flow has a structure of a pure Alfvén wave, that is $\vec{Z}_0^- = \vec{0}$ and $Z_{0_i}^+ = 2c_{A_i} + A_{ij}x_j$ is a slowly varying vector field. If we perturb this state with small amplitude fluctuations $\vec{Z}^\pm = \vec{Z}_0^\pm + \delta\vec{Z}$ (for simplicity we consider the case $|\delta\vec{Z}^+| \simeq |\delta\vec{Z}^-| = |\delta\vec{Z}|$), with $|\delta\vec{Z}| \ll |\vec{Z}_0^\pm|$ and we look for the linear stability of the structure, it can be shown [7] that an ideal instability with growth rate of the order of $\gamma_{\rm id} \sim \delta Z_\ell/\ell$ (δZ_ℓ represents an order of magnitude estimate for the characteristic perturbations at the length scale ℓ [13]) amplifies the perturbations and produces either flat (quasi-2D) or elongated (ribbonlike) structures. When these ideal MHD instabilities are saturated, the flat or ribbonlike structures appear to be unstable with respect to the usual tearing mode perturbations. These perturbations consist of filamentation of the vorticity structure, which is usually known as magnetic field reconnection. The appearance of these instabilities in numerical simulation of 2D and 3D MHD turbulence has been reported in some papers (see Refs. [6,8,7], and references therein). Actually the occurrence of tearing instabilities in MHD turbulence is a matter of different points of view. Indeed, in the above mentioned simulations there is strong evidence for the presence of tearing instabilities, while other simulations (see, for example, [5]) show that the fully turbulent state consists of an ensemble of microcurrent sheets which do not show any signs of tearing instability. In the present paper we do not enter this debate, rather we would like to emphasize some effects which could be interesting if the tearing instability is effectively at work. In fact, the occurrence of these instabilities in MHD turbulence is very interesting because, when viewed in the wave vector space, the reconnection is equivalent to a mechanism of nonlocal energy transfer towards smaller scales. This mechanism could compete with the usual nonlinear energy transfer which on the contrary is local, thus modifying the small-scale dynamic of MHD turbulence. In Ref. [7] it is shown that, if the tearing instability is at work, the usual inertial range is modified. We then argue that as a consequence of the modification of the small-scale behavior, the topological properties of the vorticity field could change at the smallest-scale length of the inertial range.

Since the typical growth rate of the stretching is of the order of the ideal times, the ideal modes may develop the flat or ribbonlike structures at times faster than those needed for the nonlinear interactions. When the instability develops, the energy is directly transferred from a length scale ℓ to a scale δ (nonlocal transfer), given by $\delta/\ell \sim S_\ell^{-1/3}$ [9], where $S_\ell \sim \delta Z_\ell \ell/\nu$ is the Reynolds number at the scale ℓ. In the presence of the velocity field the growth rate of the tearing instability is given by [9]

$$\gamma_\ell \sim S_\ell^{-1/2}\left(\frac{\delta Z_\ell}{\ell}\right) . \qquad (4)$$

The competition between the nonlinear energy cascade and the linear tearing instability consists in a modification of the energy transfer rate [7]. Indeed when γ_ℓ^{-1} is less than the nonlinear energy transfer time $\tau_\ell \sim \ell c_A/\delta Z_\ell^2$, the energy is transferred towards the smaller scales from the (nonlocal) instability rather than from the (local) nonlinear cascade. The requirement that $\gamma_\ell^{-1} < \tau_\ell$ defines a length scale

$$(\ell_t/L) \sim R^{-1/(1+3\xi_1)}$$

($R = c_A L/\nu$ is the Reynolds number of the largest scales) such that, for $\ell > \ell_t$, the transfer is driven by the nonlinear energy cascade realized on times τ_ℓ, while for $\ell < \ell_t$ the transfer is driven by the instability on times γ_ℓ^{-1}. Note that the ratio between the length ℓ_t and the dissipative length is given by $(\ell_t/\ell_d) \sim R^\beta$, where $\beta = \xi_1/(1+2\xi_1)(1+3\xi_1) \simeq 2/21$. This implies that high Reynolds numbers are required to resolve the scales under which the tearing instability eventually modifies the nonlinear energy cascade [7].

Let us consider now the vorticity $\omega = |\vec{\nabla}\times\delta\vec{Z}|$. From Eq. (1) it can be shown that the nonlinear transfer term is given by $(\vec{Z}_0^\pm\cdot\vec{\nabla})\omega$. Following Ref. [7] we can model the tearing action at the scale length ℓ, by means of a linear term given simply by $\gamma_\ell\omega$. The area of the level surface of the squared vorticity can be estimated by using the method introduced in Refs. [2,3]. This method consists essentially in determining the weighted surface area A_ℓ at a given scale ℓ, which is enclosed in a sphere of radius ℓ. Then the dimension D is found from the relation $A_\ell \sim \ell^D$. The estimate is given as an upper bound for A_ℓ by averaging over the volume of the sphere and over a short time interval δt of the order of an eddy-turnover time. The maximum value of A_ℓ is then estimated to be $A_\ell^2 \sim \ell^3(c_1\ell + G_\ell + T_\ell)$, where c_1 is a constant and G_ℓ is the advection term in Eq. (9) of Ref. [2]. This term can be estimated to be of the order of

$$\int_{\delta t} dt \int_{V_\ell} d^3\vec{x}\ \ G(\omega)(\delta\vec{Z}\cdot\vec{\nabla})\chi_\ell ,$$

where $G(\omega) \sim \delta Z_\ell^2$ is proportional to the square of the vorticity inside the sphere of volume V_ℓ, and the term $(\delta\vec{Z}\cdot\vec{\nabla})$ is the usual transfer rate. The characteristic function $\chi_\ell \simeq 1$ inside the sphere and $\chi_\ell \simeq 0$ outside the sphere. Using this term, as shown by Biskamp [4], one can find the maximum value for the function $G_\ell \sim (c_3/\nu c_A)\ell^2\delta Z_\ell^2$ (c_3 is a constant). The function T_ℓ is a new term which derives from the occurrence of the tearing instability. Since we model the instability with a linear relation [7], by using the procedure of Ref. [2] we can write

$$\int_{\delta t} dt \int_{V_\ell} d^3\vec{x}\ \ G(\omega)\gamma_\ell\chi_\ell .$$

By proceeding as in Ref. [4], that is, multiplying by $\int\chi_\ell d^3\vec{x} \sim V_\ell \sim \ell^3$ and dividing by δZ_ℓ^2, we obtain an estimate for the maximum value of the function T_ℓ. As an order of magnitude estimate, the isovorticity surfaces then turn out to be proportional to

$$A_\ell^2 \le \ell^3 \left[c_1 \ell + \frac{c_2}{\nu^{1/2}} \ell^{3/2} \delta Z_\ell^{1/2} + \frac{c_3}{\nu c_A} \ell^2 \delta Z_\ell^2 \right] \tag{5}$$

(c_2 is a constant). The first term on the right hand side of Eq. (5) derives from the dissipative terms and dominates at small scales, giving rise to smooth isovorticity surfaces with dimension $D = 2$. The third term is due to the nonlinear energy cascade realized over times τ_ℓ. It dominates at larger scales ℓ and gives rise to a dimension

$$D = \frac{11}{4} + \frac{1}{4}\left(1 - \widehat{D}_{1/2}\right) . \tag{6}$$

As shown by Biskamp [4], in the absence of intermittency the dimension $D = 11/4 = 2.75$ can be found in MHD simulations. The difference with respect to the usual estimate $D = 8/3$ in hydrodynamic flows [2,3] is due to the presence of the Alfvén effect which modifies the exponent ξ_1 from 1/3 to 1/4. The correction due to intermittency is even small. Indeed, for example, in the usual p model [15,11] where $\widehat{D}_q = \log_2 [p^q + (1-p)^q]^{1/(1-q)}$, by using the most probable value $p \simeq 0.7$ [11], it can be found that the correction is of the order of 1.6×10^{-2}, so that $D = 2.76$. The second term in Eq. (5) is due to the eventual occurrence of the resistive instabilities in MHD. This term dominates at scales $\ell_d < \ell < \ell_t$. It gives rise to a dimension of the level surface, given by

$$D = \frac{37}{16} + \frac{7}{16}\left(1 - \widehat{D}_{1/8}\right) . \tag{7}$$

In the absence of intermittency $D = 2.31$, while in the presence of intermittency $D = 2.32$. Looking at Eq. (5), it can be seen that the second term becomes dominant when the local Reynolds number $S_{\ell_*} \sim (c_1/c_2)^2$, from which we can calculate the Reynolds number dependence of the inner scale ℓ_* as

$$(\ell_*/L) \sim C_* R^{-1/(1+\xi_1)} , \tag{8}$$

with $C_* = (c_1/c_2)^{2/(1+\xi_1)}$. It can be noted that, when the reconnecting instability is not taken into account, the Reynolds number dependence of the inner scale becomes

$$(\ell_*/L) \sim C_* R^{-2/(2+\xi_1)} ,$$

where $C_* = (c_1/c_3)^{2/(2+\xi_1)}$. Finally, by looking at Eq. (8) it can be noted that the reconnecting scale ℓ_t is greater than the inner scale ℓ_* providing the ratio c_1/c_3 is small enough, that is, $(c_1/c_3) < R^\gamma$ where $\gamma = 2\xi_1/(1+3\xi_1) \simeq 2/7$.

In summary, we have investigated one of the effects of the tearing instability which, if effectively at work, could change the dynamical properties of the small-scale MHD turbulence. We argued that the instability could be responsible for the appearance, at the scaling range $\ell_d < \ell < \ell_t$, of a level set for the isovorticity with dimension of the order of $D = 37/16$. It is worthwhile to remark that the occurrence of the tearing instability in MHD turbulence is still a matter of debate, since, even if it seems to be evident in some numerical simulations, no definitive supporting evidence has been reported.

[1] K.R. Sreenivasan, R. Ramshankar, and C. Meneveau, Proc. R. Soc. London, Ser. A **421**, 79 (1989).

[2] P. Constantin, I. Procaccia, and K.R. Sreenivasan, Phys. Rev. Lett. **67**, 1739 (1991).

[3] P. Costantin and I. Procaccia, Phys. Rev. A **46**, 4736 (1992); Phys. Rev. E **47**, 3307 (1993).

[4] D. Biskamp, Europhys. Lett. **21**, 563 (1993).

[5] D. Biskamp and H. Welter, Phys. Fluids B **1**, 1964 (1989).

[6] H. Politano, A. Pouquet, and P.L. Sulem, Phys. Fluids B **1**, 2330 (1989).

[7] V. Carbone, P.L. Veltri, and A. Mangeney, Phys. Fluids A **2**, 1487 (1990).

[8] F. Malara, P.L. Veltri, and V. Carbone, Phys. Fluids B **4**, 3070 (1992).

[9] I. Hoffman, Plasma Phys. **17**, 143 (1975); M. Dobrowolny, P.L. Veltri, and A. Mangeney, J. Plasma Phys. **29**, 393 (1983); G. Einaudi, and F. Rubini, Phys. Fluids **29**, 2563 (1986).

[10] R.H. Kraichnan, Phys. Fluids **8**, 1385 (1965); M. Dobrowolny, A. Mangeney, and P.L. Veltri, Phys. Rev Lett. **45**, 144 (1980).

[11] V. Carbone, Phys. Rev. Lett. **71**, 1546 (1993).

[12] U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics,* edited by M. Gil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1984), p. 84.

[13] As is well known, the first-order structure functions $\langle Z_\ell^\pm(\vec{r})\rangle$, calculated through the differences of the longitudinal variables $Z_\ell^\pm(\vec{r}) = \vec{e} \cdot [\vec{Z}^\pm(\vec{r}+\vec{\ell}) - \vec{Z}^\pm(\vec{r})]$, represent an order of magnitude estimate for the characteristic fluctuations at the scale length ℓ. If the differences are non-normally distributed (intermittent effects) their statistical properties are defined only if we define the whole set of qth moments.

[14] H.G.E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).

[15] C. Meneveau and K.R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).